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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS WITH MULTIPLE ZEROS AND POLES

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ABSTRACT

Let $\mathcal{F}$ be a family of functions meromorphic in the plane domain D, all of whose zeros and poles are multiple. Let h be a continuous function on D. Suppose that, for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. We show that if $h(z) \neq 0$ for all $z \in D$, or if h is holomorphic on D but not identically zero there and all zeros of functions in $\mathcal{F}$ have multiplicity at least 3, then $\mathcal{F}$ is a normal family on D.

1. Introduction

In this paper, we study the normality of families of meromorphic functions on plane domains, all of whose zeros and poles are multiple. As a first result, we have

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THEOREM 1: *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D in $\mathbb{C}$, all of whose zeros and poles are multiple. Let h be a continuous function on D such that $h(z) \neq 0$ for $z \in D$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

For analytic h, this result was observed by Fang [4, Lemma 6]. As an immediate consequence, we have the

COROLLARY: *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D in $\mathbb{C}$. Suppose that for some fixed positive integer n, $f'f^n \neq 1$ on D for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family on D.*

Proof: Applying Theorem 1 to the family $\tilde{\mathcal{F}} = \{f^{n+1} : f \in \mathcal{F}\}$ with $h(z) \equiv n+1$ shows that $\tilde{\mathcal{F}}$ is normal on D. But then $\mathcal{F}$ is as well. ∎

For a discussion of the history of this last result, see [7, p. 226] and [6, pp. 18–19].

If h is allowed to vanish on D, Theorem 1 may fail, even for analytic functions h.

Example 1: Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = \frac{(z-\frac{1}{n})^2(z+\frac{1}{n})^2}{z^2} = z^2 - \frac{2}{n^2} + \frac{1}{n^4 z^2}.$$

Clearly, $\mathcal{F}$ fails to be normal in any neighborhood of 0. However, all zeros and poles of f_n are multiple; and $f_n'(z) \neq 2z$ on $\mathbb{C}$.

However, requiring that all zeros of functions in $\mathcal{F}$ have multiplicity at least 3 leads to a positive result.

THEOREM 2: *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose poles are multiple and whose zeros all have multiplicity at least 3. Let h be a function holomorphic on D, $h \not\equiv 0$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

The hypothesis that all poles are multiple cannot be omitted, as is shown by the following example.

Example 2: Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = \frac{(z-\frac{1}{n})^3}{z-\frac{3}{n}} = z^2 + \frac{3}{n^2} + \frac{8}{n^3(z-3/n)}.$$

Clearly, $\mathcal{F}$ fails to be normal in a neighborhood of 0. However, all zeros of functions in $\mathcal{F}$ have multiplicity 3; and $f_n'(z) \neq 2z$ on $\mathbb{C}$.

The plan of the paper is as follows. In Section 2, we record some known results which will be used in the proofs of Theorems 1 and 2 and prove a simple lemma on rational functions needed for those proofs. In Section 3, we prove Theorem 1. We conclude with the proof of Theorem 2 in Section 4.

2. Auxiliary results

We require the following renormalization result, which has become a standard tool in the study of normal families.

LEMMA 1 ([5, Lemma 2] cf. [7, pp. 216–217]): *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$;*
(b) *points z_n, $|z_n| < r$;*
(c) *functions $f_n \in \mathcal{F}$; and*
(d) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. In particular, g has order at most 2.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

We also require some facts about the local degree of a continuous function. See [1, p. 385] for a clear statement of the relevant facts and [3, Chapter 1] for a detailed discussion and proofs.

LEMMA 2: *Let M be the set of all triples (φ, U, w), where U is a bounded open subset of $\mathbb{C}$, $\varphi\colon \overline{U} \to \mathbb{C}$ is a continuous function, and $w \in \mathbb{C} \setminus \varphi(\partial U)$. There exists a function $d\colon M \to \mathbb{Z}$ such that*

(i) *if U is a piecewise-smoothly bounded Jordan domain and φ is holomorphic on $\overline{U}$, then $d(\varphi, U, w)$ is the winding number of $\varphi(\partial U)$ about w (and hence, by the argument principle, the number of times φ takes on the value w in U);*
(ii) *if $\psi\colon \overline{U} \to \mathbb{C}$ is a continuous function such that $|\psi(\zeta) - \varphi(\zeta)| < \mathrm{dist}(w, \varphi(\partial U))$ for each $\zeta \in \overline{U}$, then $d(\psi, U, w) = d(\varphi, U, w)$; and*
(iii) *if $d(\varphi, U, w) \neq 0$, then $\overline{U} \cap \varphi^{-1}(w) \neq \emptyset$.*

We also need the following result from value distribution theory.

LEMMA 3 ([2, Theorem 1.1]): *Let g be a transcendental meromorphic function and let R be a rational function, $R \not\equiv 0$. Suppose that all zeros and poles of g are multiple except for finitely many. Then $g' - R$ has infinitely many zeros.*

Finally, we require some facts about rational functions.

LEMMA 4 ([6, Lemma 8]): *Let f be a nonpolynomial rational function such that $f'(z) \neq 1$ for $z \in \mathbb{C}$. Then*

$$f(z) = z + c + \frac{a}{(z+b)^m},$$

where $a \neq 0$, b, and c are constants and m is a positive integer. If the zeros of f are all multiple, then $m = 1$.

LEMMA 5: (i) *Let Q be a nonconstant rational function, all of whose zeros and poles are multiple. Then $Q'(z) = 1$ has a solution in $\mathbb{C}$.*

(ii) *Let Q be a rational function, all of whose poles are multiple with the possible exception of $z = 0$ and all of whose zeros have multiplicity at least 3. Then for each positive integer k, $Q'(z) = z^k$ has a solution in $\mathbb{C}$.*

Proof: (i) If Q is a nonconstant polynomial such that $Q'(z) \neq 1$, $Q(z) = cz + d$, where $c \neq 0, 1$, and thus does not have multiple zeros. If Q is a nonpolynomial rational function all of whose zeros are multiple such that $Q'(z) \neq 1$, then by Lemma 4,

$$Q(z) = z + c + \frac{a}{z+b},$$

so that Q does not have multiple poles.

(ii) Fix k and suppose that $Q'(z) - z^k \neq 0$ for all $z \in \mathbb{C}$. If Q is a polynomial, then $Q'(z) = z^k + c$, with $c \neq 0$, so that

$$Q(z) = \frac{1}{k+1} z^{k+1} + cz + d.$$

Since all zeros of Q have multiplicity at least 3, we have $k \geq 2$ and $Q''(z) = Q'(z) = 0$ whenever $Q(z) = 0$. But $Q''(z) = kz^{k-1}$ vanishes only for $z = 0$. Thus, we must have $Q(0) = 0$, so that also $c = Q'(0) = 0$, a contradiction. Thus Q cannot be a polynomial.

Let $f(z) = Q(z) - \frac{1}{k+1} z^{k+1} + z$. Then f is a nonpolynomial rational function such that $f'(z) \neq 1$. By Lemma 4,

$$f(z) = z + c + \frac{a}{(z+b)^m}$$

so that

$$Q(z) = \frac{1}{k+1} z^{k+1} + c + \frac{a}{(z+b)^m}, \tag{2.1}$$

where $a \neq 0$, b, and c are complex numbers and m is a positive integer. Suppose that $Q(z_0) = 0$. Then since z_0 has multiplicity at least 3, we have

$$Q'(z_0) = z_0^k - \frac{ma}{(z_0+b)^{m+1}} = 0, \tag{2.2}$$

$$Q''(z_0) = k z_0^{k-1} + \frac{m(m+1)a}{(z_0+b)^{m+2}} = 0. \tag{2.3}$$

It follows from (2.2) that $z_0 \neq 0$. Solving (2.2) and (2.3) for z_0 and using $ma \neq 0$, we obtain $z_0 = -kb/(m+k+1)$. Thus $b \neq 0$, and by (2.1),

$$Q(z) = \frac{(z + \frac{kb}{m+k+1})^{m+k+1}}{(k+1)(z+b)^m}. \tag{2.4}$$

Hence, again by (2.1),

$$z^{k+1}(z+b)^m + c(k+1)(z+b)^m + a(k+1) = \left(z + \frac{kb}{m+k+1}\right)^{m+k+1}. \tag{2.5}$$

Equating coefficients of z^{m+k} in (2.5), we obtain $mb = kb$, so that $m = k$ since $b \neq 0$. Equating coefficients of z^{m+k-1} in (2.5) then shows that $k = 1$, so that $m = 1$. But this contradicts the assumption that all nonzero poles of Q are multiple. The lemma is proved. ∎

3. Proof of Theorem 1

Since normality is a local property, we may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1, there exist $f_n \in \mathcal{F}$, $z_n \in \Delta$, and $\rho_n \to 0+$ such that $g_n(\zeta) = f_n(z_n + \rho_n\zeta)/\rho_n$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g, all of whose zeros and poles are multiple. Taking a subsequence and renumbering, we may assume that $z_n \to z_0 \in \Delta$.

We claim $g'(\zeta) \neq h(z_0)$.

Clearly, $g' \not\equiv h(z_0)$, since then g would be linear and hence could not have multiple zeros. Suppose $g'(\zeta_0) = h(z_0)$. Then $\varphi = g' - h(z_0)$ is a nonconstant analytic function on a neighborhood V of ζ_0, which vanishes at ζ_0. Let $\Delta_\varepsilon = \{w : |w| < \varepsilon\}$. For $\varepsilon > 0$ sufficiently small, the component U of $\varphi^{-1}(\Delta_\varepsilon)$ containing ζ_0 is relatively compact in V and satisfies $\varphi(\partial U) = \{w : |w| = \varepsilon\}$ and

$d(\varphi, U, 0) > 0$, where d is the local degree. Set $\varphi_n(\zeta) = g_n'(\zeta) - h(z_n + \rho_n\zeta)$; then $\varphi_n \to \varphi$ locally uniformly on V. Thus, for n sufficiently large, we have $|\varphi_n(\zeta) - \varphi(\zeta)| < \varepsilon$ on $\overline{U}$. By (ii) of Lemma 2, $d(\varphi_n, U, 0) = d(\varphi, U, 0) > 0$, so that by (iii) of the same result, there exists $\zeta_1 \in \overline{U}$ such that $\varphi_n(\zeta_1) = 0$. But this contradicts $f_n'(z) \neq h(z)$ on Δ. The claim is proved.

Since $g'(\zeta) \neq h(z_0)$, it follows from Lemma 3 that g must be a rational function. But then by Lemma 5(i), g' must take on the nonzero value $h(z_0)$, a contradiction.

4. Proof of Theorem 2

By Theorem 1, it suffices to prove that $\mathcal{F}$ is normal at points for which $h(z) = 0$. So let us assume, making standard normalizations, that $\mathcal{F}$ satisfies the conditions of Theorem 2 and that

$$h(z) = z^k + a_{k+1}z^{k+1} + \cdots = z^k b(z), \quad z \in \Delta,$$

where $k \geq 1$, $b(0) = 1$, and $h(z) \neq 0$ for $0 < |z| < 1$. Consider on Δ the family $\mathcal{F}_1 = \{F = f/h : f \in \mathcal{F}\}$. If $f \in \mathcal{F}$, $f'(0) \neq h(0) = 0$; hence, since all zeros of f are multiple, $f(0) \neq 0$. Thus, for any $F \in \mathcal{F}_1$, $F(0) = f(0)/h(0) = \infty$. We shall prove that $\mathcal{F}_1$ is normal on Δ.

Suppose not. Then by Lemma 1 (with $\alpha = k = A = 1$), there exist $F_n \in \mathcal{F}_1$, $z_n \in \Delta$ ($|z_n| \leq r < 1$), and $\rho_n \to 0+$ such that

$$\frac{F_n(z_n + \rho_n\zeta)}{\rho_n} = g_n(\zeta) \to g(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic function on the plane, all of whose zeros are multiple, such that $g^{\#}(\zeta) \leq g^{\#}(0) = 2$.

We consider two cases.

(a) Suppose $z_n/\rho_n \to \infty$. Then since $g_n(-z_n/\rho_n) = F_n(0)/\rho_n$, the pole of g_n corresponding to that of F_n at 0 drifts off to infinity, and g has only multiple poles. We have

$$F_n'(z) = \frac{f_n'(z)h(z) - f_n(z)h'(z)}{h(z)^2} = \frac{f_n'(z)}{h(z)} - \frac{h'(z)}{h(z)}F_n(z).$$

Thus

$$\begin{aligned} g_n'(\zeta) = F_n'(z_n+\rho_n\zeta) &= \frac{f_n'(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} - \frac{h'(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)}F_n(z_n+\rho_n\zeta) \\ &= \frac{f_n'(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} - \Big(\frac{k}{z_n+\rho_n\zeta} + \frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\Big)F_n(z_n+\rho_n\zeta) \\ &= \frac{f_n'(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} - \Big(\frac{k}{z_n/\rho_n+\zeta} + \rho_n\frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\Big)\frac{F_n(z_n+\rho_n\zeta)}{\rho_n}. \end{aligned}$$

Clearly,

$$\lim_{n\to\infty}\frac{k}{z_n/\rho_n+\zeta}=0 \quad\text{and}\quad \lim_{n\to\infty}\rho_n\frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}=0$$

uniformly on compact sets of $\mathbb{C}$. Thus, on compact subsets of $\mathbb{C}$ disjoint from the poles of g,

$$\frac{f_n'(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} = g_n'(\zeta) + \Big(\frac{k}{z_n/\rho_n+\zeta} + \rho_n\frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\Big)g_n(\zeta)$$

converges uniformly to $g'(\zeta)$. Since $f_n'(z)/h(z)\neq 1$, by Hurwitz' Theorem either $g'\equiv 1$ or $g'(\zeta)\neq 1$ for all $\zeta\in\mathbb{C}$. The first alternative contradicts $g^{\#}(0)=2$. But if $g'\neq 1$, then by Lemma 3, g is rational; and we obtain a contradiction to Lemma 5(i).

(b) So we may assume that $z_n/\rho_n\to\alpha$, a finite complex number. We have

$$\frac{F_n(\rho_n\zeta)}{\rho_n} = \frac{F_n(z_n+\rho_n(\zeta-z_n/\rho_n))}{\rho_n} \to g(\zeta-\alpha)=\tilde{g}(\zeta),$$

the convergence being spherically uniform on compact sets of $\mathbb{C}$ and hence uniform on compacta disjoint from the poles of $\tilde{g}$. Clearly, all zeros of $\tilde{g}$ have order at least 3 and all poles are multiple except possibly the pole at 0, which has order at least k.

Now

$$\lim_{n\to\infty}\frac{h(\rho_n\zeta)}{\rho_n^k}=\zeta^k$$

uniformly on compact subsets of $\mathbb{C}$. Thus, writing

$$G_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^{k+1}} = \frac{h(\rho_n\zeta)}{\rho_n^k}\frac{f_n(\rho_n\zeta)}{\rho_n h(\rho_n\zeta)} = \frac{h(\rho_n\zeta)}{\rho_n^k}\frac{F_n(\rho_n\zeta)}{\rho_n},$$

we have

$$G_n(\zeta)\to\zeta^k\tilde{g}(\zeta)=G(\zeta)$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of $\tilde{g}$. Note that since $\tilde{g}$ has a pole of order at least k at 0, $G(0)\neq 0$.

We claim that $G'(\zeta) \neq \zeta^k$. Indeed, suppose that $G'(\zeta_0) = \zeta_0^k$. Then G is holomorphic at ζ_0 and

$$G_n'(\zeta) - \frac{h(\rho_n\zeta)}{\rho_n^k} = \frac{f_n'(\rho_n\zeta) - h(\rho_n\zeta)}{\rho_n^k} \neq 0.$$

Thus, if $\zeta_0 \neq 0$, we have $G'(\zeta) \equiv \zeta^k$ by Hurwitz' Theorem and hence $G(\zeta) = \zeta^{k+1}/(k+1) + C$. Since all zeros of G are multiple, $C = 0$. But then $\tilde{g}(\zeta) = \zeta/(k+1)$, which contradicts the fact that $\tilde{g}$ has a pole at 0.

The same argument applies if $\zeta_0 = 0$. Indeed, in that case, G is analytic at 0, so $\tilde{g}$ has a pole of exact order k at 0. Since for each n, the pole of $F_n(\rho_n\zeta)$ at 0 has order k, it follows that there exists $\delta > 0$ such that $F_n(\rho_n\zeta)$ has no poles in $\Delta_\delta' = \{z : 0 < |z| < \delta\}$. Thus G_n is holomorphic on $\Delta_\delta = \{z : |z| < \delta\}$, so $G_n \to G$ uniformly on a neighborhood of 0 as well. We may then apply Hurwitz' Theorem as above.

Thus $G'(\zeta) \neq \zeta^k$. It follows from Lemma 3 that G must be a rational function. However, then Lemma 5(ii) shows that $G'(\zeta) = \zeta^k$ has a solution in $\mathbb{C}$. The contradiction establishes that $\mathcal{F}_1$ is normal on Δ.

It remains to show that this implies that $\mathcal{F}$ is normal on Δ. Since $\mathcal{F}_1$ is normal on Δ (and hence, as a collection of maps from Δ to $\hat{\mathbb{C}}$, equicontinuous on compacta) and $F(0) = \infty$ for each $F \in \mathcal{F}_1$, there exists $\delta > 0$ such that if $F \in \mathcal{F}_1$, then $|F(z)| \geq 1$ for $z \in \Delta_\delta$. Hence $f(z) \neq 0$ for $z \in \Delta_\delta$ for all $f \in \mathcal{F}$. Now since $h(z) \neq 0$ for $z \in \Delta_1'$, $\mathcal{F}$ is normal on Δ_1' by Theorem 1. Suppose that $\mathcal{F}$ is not normal on Δ_δ. Then there exists a sequence $\{f_n\} \subset \mathcal{F}$ which converges spherically uniformly on compact subsets of Δ_δ', but none of whose subsequences converges spherically uniformly on a neighborhood of 0. By the invariance of the spherical metric, the same holds for the sequence $\{1/f_n\}$, whose members are all holomorphic on Δ_δ. It follows (by the maximum modulus principle) that $\{1/f_n\}$ diverges uniformly to infinity on compact subsets of Δ_δ'. Thus $\{f_n\}$ converges uniformly to 0 on compact subsets of Δ_δ' and hence so does $\{F_n\}$, where $F_n = f_n/h$. But $|F_n(z)| \geq 1$ for $z \in \Delta_\delta$, since $F_n \in \mathcal{F}_1$. The contradiction shows that $\mathcal{F}$ is normal on Δ_δ and hence on $\Delta = \Delta_\delta \cup \Delta_1'$. This completes the proof of Theorem 2.

Remark: In the proofs of Theorem 1 and case (a) of Theorem 2, we could have invoked Theorem 1 (or Lemma 9) of [6] in place of the combination of Lemma 3 and Lemma 5(i) above.

With only the slightest modifications, the proof of Theorem 2 also yields the following result.

THEOREM 3: *Let $\mathcal{F}$ be a family of functions meromorphic on domain D in $\mathbb{C}$, all of whose zeros all have multiplicity at least 4. Let h be a function holomorphic on D, $h \not\equiv 0$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

Details are left to the reader.

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